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UNIT-I

* **Norm:-** A norm on X is real-valued function $\|\cdot\|: X \to R$ or $[0, \infty)$ defined on X such that for any $x, y \in X$ and for all $\lambda \in K$

(i) $\|x\| \geq 0$
(ii) $\|\lambda x\| = |\lambda| \|x\|$
(iii) $\|x+y\| \leq \|x\| + \|y\|$
(iv) $\|x\| = 0 \Leftrightarrow x = 0$

Note:- ① Norm is a distance function, norm is unary function
② matric is also a distance function and binary function.

* **Norm linear Space:-**
Norm on a vector space X is a real function $\|\cdot\|: X \to R$, defined on X is called norm linear space; such that for any $x, y \in X$ and $\forall \lambda \in K$.

(i) $\|x\| \geq 0$
(ii) $\|x+y\| \leq \|x\| + \|y\|$
(iii) $\|\lambda x\| = |\lambda| \|x\|$
(iv) $\|x\| = 0 \Leftrightarrow x = 0$

* **Semi Norm:-** If the property for four of the norm not satisfy then

it is called semi norm

OR A semi norm on X is a real-valued function $\|\cdot\|: X \to R$ such that

(i) $\|x+y\| \le \|x\| + \|y\| \quad \forall x, y \in X$ and

(ii) $\|\lambda x\| = |\lambda| \|x\|$

where λ is scalar and $x \in X$.

* P-norm:

A p-norm $(p>0)$ is a real valued function $\|\cdot\|: X \to R$ defined on X such that:

(P-1) $\|x\| = 0 \Leftrightarrow x = 0$

(P-2) $\|x+y\| \le \|x\| + \|y\|$

(P-3) $\|\lambda x\| = |\lambda|^p \|x\|$.

if $\|\cdot\|$ is a p-norm on X, then the pair $(X, \|\cdot\|)$ is a p-normed space. Usually we take $0<p\le 1$. A p-norm with $p=1$ is just a norm.

Examples on normed linear space

(1) R^n is a nls with the norm defined by

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

(2) for $1 \le p < \infty$, l_p is a nls with the norm defined by

$$\|x\| = \left(\sum_{k} |x_k|^p\right)^{1/p}$$

* Bounded linear Transformation :-

let X and Y be two nls over field K then the mapping $T: X \to Y$ is called bounded linear transformation if.

(i) T is linear.

(ii) T is bounded $\exists$ $M>0$ s.t.

(iii) $\|Tx\| \le M\|x\| \quad \forall x \in X$.

* Bounded linear Operators :-

let X be a nls then bounded linear transformation from X to X is called bounded linear operator denoted by $B(X)$.

$B(X)$ set of all bounded linear operator.

* Bounded linear Function :-

let X be a nls over the field K the bounded linear transformation from X to K is called bounded linear function

$T: X \to K$

X^* set of all bounded linear function of X.

* Continuous Transformation (Pointwise continuous transformation):- A linear transformation $T: X \to Y$ is said to be

continuous at $x_0 \in X$ if for each $\varepsilon > 0$ then $\exists\ \delta > 0$ (depending on $\varepsilon \in x_0$)

$$\|Tx - Tx_0\| < \varepsilon$$

and $\|x - x_0\| < \delta$

* **Uniform Continuity :-**

The function f is uniformly continuous on I (interval) if for every $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$|x-y| < \delta$$

implies

$$|f(x) - f(y)| < \varepsilon$$

* **Category :-**

let X be a metric space and M be a subset of X then M is said to be

① Rare (No where dense) in X if $\mathrm{int}(\overline{M}) = \phi$

② First category in X if M is the union of countably many sets each of which is rare in X

③ Second category in X if M is not of first category in X.

* **Baires Category Theorem :-**

If a metric space X complete it is of second category in itself. Hence if $X \neq \phi$ is a complete metric space and

$$X = \bigcup_{k=1}^{\infty} A_k \quad (A_k \text{ is closed})$$

Then atleast one A_k contains are non-empty set.

OR every complete metric space is a second category.

5, 11, 12, 14, 16 * **Uniform Boundedness Theorem**

(Banach - Steinhaus Theorem)

Statement :- let $\{T_i\}$ be a non-void family of bounded linear transformation from a Banach space X into a norm linear space Y. If $\sup_i \|T_i(x)\| < \infty$ for each $x \in X$, Then, $\sup_i \|T_i\| < \infty$

OR If $\{T_i(x)\}$ is bounded for every $x \in X$ then $\{\|T_i\|\}$ is bounded.

(page - 184) H.K.

Proof:- for each positive integer n, let we define.

$F_n = \{x : x \in X \text{ and } \|T_i x\| \le n \ \forall i\}$

Then F_n is closed subset of X for this $x \in \overline{F_n}$ then there is a sequence $\{x_j\}$ in F_n convergent to X this means that for every fixed i.

$\|T_i x_j\| \le n$

$\Rightarrow \|T_i x\| \le n.$

Hence $x \in F_n$ and F_n is closed.

Now, for each $x \in X$, belong to same F_n.

Hence, $X = \bigcup_{n=1}^{\infty} F_n$

If not then $x \in X \ = x \notin \bigcup_{n=1}^{\infty} F_n$

$\Rightarrow x \notin F_n$ for each n.

$\Rightarrow \|T_i(x)\| >$ for all i.

which is a contradiction of our assumption

$X = \bigcup_{n=1}^{\infty} F_n$

since X is complete by Baire category theorem are of Fn's.

say F_{n_0} has non-empty integer this F_{n_0} contains and open ball with centre x_0 and radius y_0.

i.e. $B_0(x_0, y_0) \subset F_{n_0}$

let $x \in X$ be arbitrary not zero.

$z = x_0 + y_0$

where

$y = \frac{x_0}{2\|x\|}$

Then $\|z - x_0\| = y\|x\| = \frac{x_0}{2\|x\|} . \|x\| < y_0$

so, that, $z = B_0 \subset F_{n_0}$

Therefore, by the def^n of F_n, we does have

$\|T_i x\| \le n_0$ for all i

also,

$\|T_i x_0\| \le n_0$

since, $x \in B_0 \subset F_{n_0}$

does from above,

$x = \frac{z - x_0}{y}$ for all i

$$\|T_i(x)\| = \frac{1}{y}\|T_i(z-x_0)\|$$

$$\le \frac{1}{y}(\|T_i(z)\| - \|T_i(x_0)\|)$$

$$\le \frac{2\|x\|}{y_0}(n_0+n_0)$$

$$\le \frac{4n_0\|x\|}{y_0}$$

$$< \frac{4n_0\|x\|}{y_0} = 1$$

Hence, for all i,

$$\|T_i\| = \sup_{\|x\|=1}\{\|T_i(x)\|\}$$

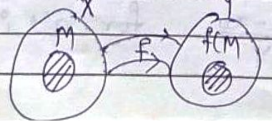
$$< \frac{4n_0}{y_0}$$

$$\Rightarrow \|T_i(x)\| < \infty$$

$$\Rightarrow \boxed{\|T_i\|<\infty}$$

Hence Proved.

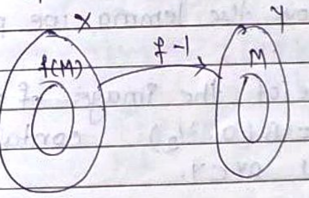
* Open mapping: let X and Y be two n/s over the field K (real and complex) then a mapping $f: X \to Y$ is said to be open mapping if image of every open set in X is open in Y.



OR let X and Y be normed linear spaces and let $T: X \to Y$ be a linear transformation. Then T is said to be an open map if T maps open sets of X into open sets of Y.

T is said to be a closed map if it maps closed sets into closed sets.

* Continuous mapping: A mapping $f: X \to Y$ is said to be continuous mapping of inverse image of open set in Y is open in X.



* Bi-continuous: A function $f: X \to Y$ is said to be Bi-continuous if f is open as well as continuous.

13 15 9 10 * The open mapping Theorem

Statement: let X and Y be Banach spaces, and T be a continuous linear transformation of X onto Y.

Then T is an open mapping.

Proof:- To prove above the find of all we solve prove following lamma.

14) Lemma:- A bounded linear transformation T from a banach space X into a banach space Y has the property that image $T(B_0)$ of the open unit ball

$$B_0 = B(0,1) \subset X$$

contain an open ball about $0 \in Y$

Proof:- Prove the lemma, we prove that:-

(a) The closure of the image of the open ball $B_1 = B(0, 1/2)$ contains an open ball $B^* \subset Y$.

(b) $T(B_n)$ contains an open ball V_n about $0 \in Y$ where;

$$B_n = B(0, 2^{-n}) \subset X$$

(c) $T(B_0)$ contains an open ball about $0 \in Y$

Proof of (a):- let $A \subset X$ we shall define

$$\alpha A = \{x : x = \alpha a, a \in A\}, x \in X \quad \text{——(I)}$$

$$A + w = \{x \in X, x = a + w, a \in w\} \quad \text{——(II)}$$

and similarly for subset of Y, we consider the open ball

$$B_1 = B(0, \tfrac{1}{2}) \subset X$$

let any fixed $x \in X$ is in kB_1 with real k. $(k > 2\|x\|)$ where,

$$X = \bigcup_{k=1}^{\infty} kB_1$$

since T is onto and linear. i.e.

$$Y = T(X) = T\left(\bigcup_{k=1}^{\infty} kB_1\right) \quad \text{——(III)}$$

$$= \bigcup_{k=1}^{\infty} (kT(B_1)) \qquad X \xrightarrow{T} T(X)$$

$$= \bigcup_{k=1}^{\infty} \left(\overline{kT(B_1)}\right)$$

since, Y is complete, it is of 2nd category.

Hence, by Banach category Theorem $\overline{kT(B_1)}$ closure must contain same open ball for since k.

$\Rightarrow$ That $\overline{T(B_1)}$ contains also an open ball.

say,

$B^* = B(y_0, \varepsilon) \subset \overline{T(B_1)}$

$B^* - y_0 = B(0, \varepsilon) \subset \overline{T(B_1)} - y_0$ —— (iv)

Proof of (b):- Now we prove that

$B^0 - y_0 \subset \overline{T(B_0)}$ this we to by showing that

$\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}$ —— (v)

let $y \in \overline{T(B_1)} - y_0$ then, $y + y_0 \in \overline{T(B_1)}$ and also, $y_0 \in \overline{T(B_1)}$

$u_n = T(w_n) \in T(B_1)$ such that $u_n \to y + y_0$ and

$v_n = T(z_n) \in T(B_1)$ such that $v_n \to y_0$

since, $w_n, z_n \in B_1$ and B_1 has radius $1/2$ it follows that,

$\|w_n - z_n\| \leq \|w_n\| + \|z_n\|$

$< \frac{1}{2} + \frac{1}{2} = 1$

$\Rightarrow w_n - z_n \in B_0 = B(0,1)$

Now,

$T(w_n - z_n) = T(w_n) - T(z_n)$

$= u_n - v_n$

$= y + y_0 - y_0 = y$

Thus we see that $y \in \overline{T(B_0)}$ since $y \subset \overline{T(B_1)} - y_0$ arbitrary

$\therefore \overline{T(B_1)} - y_0 \subset \overline{T(B_0)}$

from eqn (iv) we get.

$B^* - y_0 = B(0, \varepsilon) \subset \overline{T(B_0)}$ —— (vi)

let

$B_n = B(0, 2^{-n}) = B(0, \frac{1}{2^n}) \subset X$

since T is linear,

$\overline{T(B_n)} = 2^{-n}\, T(B_0)$

from eqn (vi) we get,

$V_n = B(0, \frac{\varepsilon}{2^n}) \subset \overline{T(B_n)}$ —— (vii)

Proof of (c):- we finally prove that

$V_1 = B^0(0, \varepsilon/2) \subset \overline{T(B_1)}$

By showing that every $y \in V_1$ in $T(B_n)$ show that,

$y \in V_n$ from eqn (vii)
with $n=1$ we have,

$$V_1 \subset \overline{T(B_1)}$$

so, that there must be $v \in T(B_1)$ choose to y (say)

$$\|y - v\| < \frac{\varepsilon}{2^n}$$

Now, $v \in T(B_1)$ implies
$$v = T(x_1), \quad x_1 \in B_1$$

Hence,
$$\|y - T(x_1)\| < \frac{\varepsilon}{2^2}$$

$$\|y - T(x_1)\| < \frac{\varepsilon}{4}$$

from eqn (vii) with $n=2$, we see that

$$y - T(x_1) \in V_2 \subset \overline{T(B_2)}$$

we calculate that where is an $x_2 \in B_2$ such that,
$$\|y - T(x_1) - T(x_2)\| < \frac{\varepsilon}{8}$$

Hence,
$$y - T(x_1) - T(x_2) \in V_3 \subset \overline{T(B_3)}$$

cauchy sequence → convergence → complete → Banach space.

in the n^{th} step we choose an $x_n \in B_n$ such that

$$\|y - \sum_{k=1}^{n} T x_k\| < \frac{\varepsilon}{2^{n+1}}$$

let $z_n = x_1 + x_2 + \dots + x_n$
since $x_n \in B_n$ where $\|x_k\| < \frac{1}{2^k}$

$$\Rightarrow \|z_n - z_m\| \leq \sum_{k=m+1}^{n} \|x_k\|$$

$$< \sum_{k=m+1}^{n} \frac{1}{2^k} \to 0 \text{ as } m \to 0$$

since $\{z_n\}$ is a cauchy sequence,
Thus, $\{z_n\}$ convergence say $z_n \to x \in X$
since X is complete.
since B_0 has radius 1

$$\sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} 1/2^k \quad \text{——— (ix)}$$

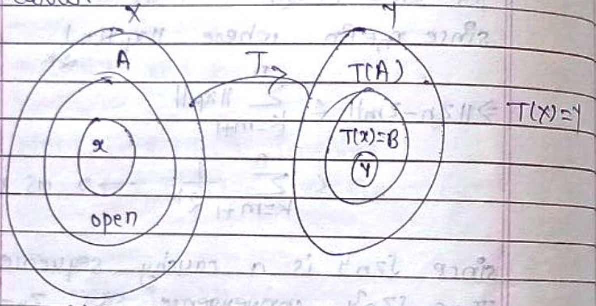
$$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \quad \left\{\because \frac{a}{1-r}\right\}$$

$$= \frac{1/2}{1 - 1/2} = 1$$

$\therefore x \in B_0$ since T is continuous.
$\therefore y \in T(B_0)$

Proof of the main Theorem :-

we prove that every open set $A \subset X$ the image $T(A)$ is open in Y this we have showing that for every $y \in Tx \in T(A)$ contain in open ball about.



T(A) open.

let $y \in T(A)$, $y \in B \subset T(A)$

let $y = T(x) \in T(A)$

since A is open therefore contains in open ball with centre x and $A(x)$. contains in open ball centre $0 \in X$.

let the radius of the ball r and say $K = \frac{1}{r_0}$

Then $K(A-x)$ contain in open unit ball $B(0,1)$

Note :- $\{ B(x,r) \subset A$

$B(x,r) - x \subset A - x$

$\Rightarrow B(0,r) \subset A - x$

$\Rightarrow \frac{1}{r} B(0,r) \subset \frac{1}{r}(A-x)$

$\Rightarrow B(0,1) \subset K(A-x) \}$

above lemma this implies that

$T(K(A-x)) = K(TA - Tx)$ {$\because$ T is linear bounded}

$= B(0,d) \subset TA - Tx$

$= Tx - 1\, B(0,d) \subset TA$

$= B(Tx, d) \subset TA$

$= B(Y, d) \subset TA$

contains an open ball about $0 \in Y$ and so does $TA - Tx$.

$\because$ TA contains an open about $Tx - Y$.

since $y \in TA$ is arbitrary and y is an interior point of TA.

Hence, $T(A)$ is open.

consequently, T is open mapping.

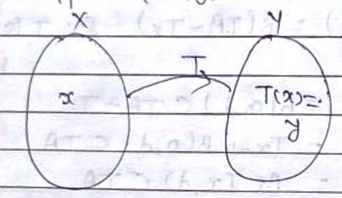
Hence Proved.

$\ast$ Closed Graph

Graph of transformation :-

let X and Y be two nls and T be a transformation from X into Y then the graph of T is denoted by G_T and defined as,

$$G_T = \{(x,y) : x \in X \text{ and } y \in Tx \in Y\}$$



$x \in X$
$y = Tx \in Y$

G_T is closed thus graph of T i.e. $G_T \subseteq X \times Y$.

(i) The graph of T is said to be closed if G_T is closed in $X \times Y$.

(ii) bijective (one-one, onto) then T^{-1} is continuous.

(iii) G_T is closed graph then T is closed mapping.

10, 13, 15 $\ast$ Closed Graph Theorem :-

Statement :- let X and Y be banach space and let T be linear transformation from X into Y then T is continuous mapping if and only if its graph G_T is closed.

Proof :- We have $B(X,Y)$ is defined as set of all bounded linear transformation from X to Y.

let $G_T = \{(x,y) : x \in X \text{ and } y \in Tx \in Y\}$ be a closed graph in $X \times Y$.

we define norm in $X \times Y$ by,

$$\|(x,y)\| = \|x\| + \|y\| \quad \text{——— (1)}$$

first we shall show that $X \times Y$ with norm define Y eqn (1) complete for this.

let $\{z_n\}$ be a cauchy sequence in $X \times Y$ where,

$$z_n = (x_n, y_n) \quad , x_n \in X \text{ and } y_n \in Y.$$

This for each $\varepsilon > 0$ $\exists$ a natural number N such that,

$$\|z_n - z_m\| < \varepsilon \quad \text{for } m,n \geq N$$

This implies that,

$$\|z_n - z_m\| = \|(x_n, y_n) - (x_m, y_m)\| < \varepsilon \ \forall \ n,m \geq N$$

$\Rightarrow \|z_n - z_m\| = \|x_n - x_m, y_n - y_m\| < \varepsilon \quad \forall n, m \geq N$

$\Rightarrow \|z_n - z_m\| = \|x_n - x_m\| + \|y_n - y_m\| < \varepsilon \quad \forall n, m \in N$ —— (2)

Hence, $\{x_n\}$ and $\{y_n\}$ are cauchy sequence in X and Y. respectively. on Y convergent $x_n \to x$ and $y_n \to y$ because X and Y are complete.

This implies that $z_n \to z = (x, y)$

since from eqn (2) with letting $m \to \infty$ we have.

$\|z_n - z\| < \varepsilon \quad \forall n \geq N$

with shows that cauchy sequence $\{z_n\}$ in $X \times Y$. convergent to $z \in X \times Y$. since $\{z_n\}$ was arbitrary.

$\therefore X \times Y$ is complete by assumption G_T is closed in $X \times Y$.

Therefore G_T is complete because closed subset of a complete space is complete.

Now, we consider the mapping $W : G_T \to X$ such that $W(x, Tx) = x$

Then W is linear and bounded.

(i) W is linear :- let $\alpha, \beta \in K$ (field) and $(x_1, Tx_1), (x_2, Tx_2) \in G_T$.

Now,

$= W[\alpha(x_1, Tx_1) + \beta(x_2, Tx_2)]$

$= W[(\alpha x_1, \alpha Tx_1) + (\beta x_2, \beta Tx_2)]$

$= W(\alpha(x_1), \alpha Tx_1) + W(\beta(x_2), \beta Tx_2)$

$= \alpha W(x_1, Tx_1) + \beta W(x_2, Tx_2)$

$\Rightarrow W$ is linear.

(ii) W is well define :- let

$(x_1, Tx_1) = (x_2, Tx_2)$

$\Rightarrow x_1 = x_2$

$\Rightarrow W(x_1, Tx_1) = W(x_2, Tx_2)$

$\Rightarrow W$ is well define.

(iii) W is bounded :- Therefore

$\|W(x, T(x))\| \leq \|x\| + \|T(x)\|$

$< \|x, T(x)\|$

$\therefore W$ is bounded.

(iv) W is bijective :- In fact the inverse mapping $W^{-1} : X \to G_T$ such that

$W(x) = (x, Tx)$

since X and G_T are complete and W is bijective.

Thus, by open mapping theorem W^{-1} is bounded.

say. $\|W^{-1}(x)\| = \|x, T(x)\| \leq b\|x\|$ for some $b>0$ & $x \in X$.

This shows that, T is bounded.

$\Rightarrow T \in B(X, Y)$

Hence Proved.

2016 * **Theorem:-** Let T be a closed linear map of a Banach space X into a Banach space Y. Then T is continuous.

Proof:- Let X and Y be Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. Then for $1 \leq p \leq \infty$, the product $X \times Y$ with the norm

$$\|(x, y)\|_p = \begin{cases} (\|x\|_1^p + \|y\|_2^p)^{1/p} & \text{if } 1 \leq p < \infty \\ \max\{\|x\|_1, \|y\|_2\} & \text{if } p = \infty \end{cases}$$

for $(x, y) \in X \times Y$, is a Banach space. define

$$G_T = \{(x, T(x)) : x \in X\}$$

$\Rightarrow T$ is closed, it follows that G_T is a closed subspace of the Banach space $X \times Y$ with the norm induced by $\|\cdot\|_p$.

Now, we define the projections

$W_1 : G_T \to X$ by $W_1[(x, T(x))] = x$

and

$W_2 : G_T \to Y$ by $W_2[(x, T(x))] = T(x)$.

for all $(x, T(x)) \in G_T$.

By the definitions of W_1 and W_2 it is evident that,

$W_1^{-1}(X) = G_T$

and

$W_2^{-1}((T(X)) = G_T$

where X, $T(X)$ and G_T are closed, it follows that W_1 and W_2 are both continuous maps from G_T onto X and into Y respectively.

Hence, by open mapping theorem W_1 is

an open mapping. also W_1 is one-one.

if $(x, T(x)), (x', T(x')) \in G_T$ such that,

$W_1[(x, T(x))] = W_1[(x', T(x'))]$.

Then we have,

$x = x'$

and so,

$T(x) = T(x')$

i.e. $(x, T(x)) = (x', T(x'))$

$\Rightarrow W_1$ is open, it follows that W_1^{-1} is continuous and that,

$W_2 W_1^{-1}(x) = W_2[(x, T(x))] = T(x) \quad \forall x \in X.$

$\Rightarrow W_2 W_1^{-1} = T$

$\therefore$ T is a composition of two continuous mapping W_2 and W_1^{-1}.

$\Rightarrow$ T is continuous.

* Theorem:- A closed linear map T mapping a normed linear space X of the second category into a Banach space Y is continuous.

Proof:- let $M = \{x \in X : T(x) = 0\}$.

Then M is a closed subspace of X since T is closed.

if $X = M$, then the proof is trivial.

Now, we suppose that $X \neq M$.

$\Rightarrow$ X is of the second category and M is a closed subspace of X, it follows that the quotient space X/M is of second category.

Now, we define $T_0 : X/M \longrightarrow Y$ by.

$T_0(\phi(x)) = T(x)$

where ϕ is the natural mapping from X onto X/M defined by

$\phi(x) = x + M \quad \forall x \in X.$

we observe that:-

① T_0 is well-defined:- if $x, x' \in X$ such that

$\phi(x) = \phi(x')$ then,

$\phi(x) = \phi(x') \Rightarrow x + M = x' + M$

$\Rightarrow x - x' \in M$

$\Rightarrow T(x - x') = 0$ {by def^n of M}

$\Rightarrow T(x) - T(x') = 0$

{$\because$ T is linear}

$\Rightarrow T(x) = T(x')$

$\Rightarrow T_0(\phi(x)) = T_0(\phi(x'))$.

$\Rightarrow T_0$ is well-defined.

(ii) T_0 is one-to-one :- if $x, x' \in X$ such that $T_0(\phi(x)) = T_0(\phi(x'))$, then,

$T_0(\phi(x)) = T_0(\phi(x')) \Rightarrow T(x) = T(x')$

$\Rightarrow T(x) - T(x') = 0$

$\Rightarrow T(x - x') = 0$

$\Rightarrow x - x' \in M$

$\Rightarrow x + M = x' + M$

$\Rightarrow \phi(x) = \phi(x')$

$\Rightarrow T_0$ is one-one.

(iii) T_0 is a closed map :- since T is closed.

(iv) T_0 is linear :- if $x, x' \in X$ and λ scalar, then

$T_0(\lambda\phi(x) + \mu\phi(x')) = T_0(\lambda(x+M) + \mu(x'+M))$

$\Rightarrow T_0((\lambda x + M) + (\mu x' + M))$

$\Rightarrow T_0((\lambda x + \mu x') + M)$

$\Rightarrow T_0(\phi(\lambda x + \mu x'))$

$\Rightarrow T(\lambda x + \mu x')$

$\Rightarrow \lambda T(x) + \mu T(x')$

$\Rightarrow \lambda T_0(\phi(x)) + \mu T_0(\phi(x'))$.

$\Rightarrow T_0$ is linear.

Thus, T_0 is well defined, one-one, closed and linear, Hence the mapping

$T_0^{-1} : T(X) \rightarrow X/M$.

is closed where $T(X) \subset Y$.

$\Rightarrow T_0^{-1}$ is open and T_0 is continuous.

14 **Theorem :-** let N and N' be normed linear space and $D \subset N$. Prove that a linear transformation $T : D \rightarrow N'$ is closed if and only if G_T is closed.

Proof :- suppose $T : D \rightarrow N'$ is a closed linear transformation. To show that the graph G_T is closed.

let (x, y) be a limit point of G_T, then $\exists$ a sequence of points $\{(x_n, T(x_n))\}$ in G_T converging to (x, y) then we get.

$\|(x_n, T(x_n)) - (x, y)\| \rightarrow 0$ as $n \rightarrow \infty$. — (1)

Now,

$\|(x_n, T(x_n)) - (x, y)\| = \|(x_n - x, T(x_n) - y)\|$

$= \|x_n - x\| + \|T(x_n) - y\|$ — (2)

using eqn (1) and (2) we get,

$\|x_n - x\| \to 0$ and $\|T(x_n) - y\| \to 0$ as $n \to \infty$

This shows this

$x_n \to x$ and $T(x_n) \to y$ as $n \to \infty$

$\Rightarrow$ T is closed, we get $x \in D$ and $T(x) = y$.

Thus, $(x, y) \in G_T$

$\Rightarrow G_T$ is closed.

<u>Conversely</u>:- suppose the graph G_T is closed. Then we have to show that T is closed linear transformation.

let $x_n \in D$, $x_n \to x$ and $T(x_n) \to y$.

Then, every neighbourhood of (x, y) contains a point of G_T.

$\Rightarrow (x, y) \in \overline{G_T}$ since G_T is closed.

$\overline{G_T} = G_T$

$\Rightarrow (x, y) \in G_T$

$\Rightarrow$ It follows by the definition of G_T that $x \in D$ and $y = T(x)$

$\Rightarrow$ T is a closed transformation.

* <u>Dense set</u>:- let (X, T) be a topological space then a subset A of X is said to be dense in X, if and only if $\overline{A} = X$. where $\overline{A}$ has empty interior. X is called seperable if X contains a countable dense subset.

Exa ① set of integer $\mathbb{Z}$ is nowhere dense in R

② set of rational no. Q is dense in R and Q is countable, Therefore R is seprable with the usual topology.

③ l_p space is seperable.

* <u>first Category / second Category</u>:- let (X, d) be a metric space $A \subseteq X$. possible the whole space is said to be first category, if it is the union of countable family of nowhere dense set, otherwise it is said to be of second category.

* <u>Closed linear Transformation</u>:- let X and Y be nls and let A be a subspace of X. Then a linear transformation $T: A \to Y$ is said to be closed iff $x_n \in A$ & $x_n \to x$

$\Rightarrow x \in A$ and $T(x) = y$.

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* **Theorem:**— let T be a bounded linear transformation from a banach space X onto nls Y. then the openness of T implies completeness of Y.

Proof:— let $\{y_n\}$ be a cauchy sequence in Y. then we can find a sequence $\{n_k\}$ of positive integers s.t. $n_k < n_{k+1}$ and for each k:

$\|y_{n_{k+1}} - y_{n_k}\| < \frac{1}{2^k}$

since T is open then $\exists N>0$ s.t.

$T(x_k) = y_{n_{k+1}} - y_{n_k}$

and,

$\|x_k\| \le N \cdot \|y_{n_{k+1}} - y_{n_k}\|$

Thus, $\sum_{k=1}^{\infty} \|x_k\| < \infty$

since X is complete. $\exists\, x \in X$.

$x = \lim_{n\to\infty} \sum_{k=1}^{n} x_k$

also since T is continuous and linear

$\sum_{k=1}^{n} T(x_k) \to T(x)$

Thus, shows that

$y_{n_{k+1}} - y_{n_1} \to T(x)$

since $\{y_n\}$ is a cauchy sequence it follows that

$y_n \to y_{n_1} + T(x)$.

Thus, $\{y_{n_k}\}$ is a convergent subsequence of the cauchy sequence $\{y_n\}$

$\Rightarrow Y$ is complete.

* **Theorem:**— let T be a bounded linear map from a nls X onto a nls Y. Then T is open iff there is $d>0$ s.t. for each $y \in Y$ there is $x \in X$ where, $Tx = y$ and $\|x\| \le d\|y\|$.

Proof:— let T be a bounded linear map from a nls X onto a nls Y.

suppose T is open. clearly $T^{-1}: Y \to X$ is continuous.

$\therefore T$ is linear then T^{-1} is linear thus, T^{-1} is bounded linear map from Y to X.

let $y \in Y$

since T is onto
$\therefore \exists\ x \in X$ s.t. $Tx = y$
$\Rightarrow x = T^{-1}y$ and
$\|x\| = \|T^{-1}y\|$
$\leq \|T^{-1}\|\,\|y\|$
boundedness of $T^{-1} \Rightarrow \exists\ \lambda > 0$ s.t. $\|T^{-1}\| \leq \lambda$
$\therefore \|x\| \leq \lambda\|y\|$

Conversely :- let there is $\lambda > 0$ s.t. for each $y \in Y$. there is $x \in X$ where $Tx = y$ and $\|x\| \leq \lambda\|y\|$.
Then we have to show that T is open

for this let G be an open set in X.
let $y \in T(G)$, $\exists\ x \in G$ s.t. $Tx = y$

$\because G$ is an open set $\exists$ an open sphere $S_\delta(x)$ in X s.t. $S_\delta(x) \subset G$.
Now consider an open sphere $S_{\delta/\lambda}(y)$ in Y.
we claim that $S_{\delta/\lambda}(y) \subseteq T(G)$
for this let $w \in S_{\delta/\lambda}(y)$ Then $\exists\ z$ s.t. $Tz = w$
Now
$\|z - x\| \leq \lambda\|T(z-x)\|$
$= \lambda\|Tz - Tx\| = \lambda\|w - y\|$
$< \lambda \cdot \frac{\delta}{\lambda} = \delta$
i.e. $\|z - x\| < \delta$

$\Rightarrow z \in S_\delta(x) \subset G$
$\Rightarrow z \in G$
$\Rightarrow T(z) \in T(G)$
$\Rightarrow w \in T(G)$
$w \in S_{\delta/\lambda}(y) \Rightarrow w \in T(G)$
$\therefore S_{\delta/\lambda}(y) \subseteq T(G)$
$\Rightarrow y$ is an interior point of $T(G)$
$\Rightarrow T(G)$ is open.
$\Rightarrow T$ is open

* Theorem :- continuity implies closedness but closedness may not be implies continuity.

Proof :- let X and Y be nls and T is transformation from X into Y.
suppose T is continuous
let $x_n \to x$ and $Tx_n \to y$ in Y.
since T is continuous therefore $Tx_n \to Tx$ in Y. but limit of convergence sequence is unique therefore $Tx = y$.
$\Rightarrow T$ is closed.
i.e. continuity implies closedness

for its conversely :- let $X = Y = R$ then $T : X \to Y$ defined by

$$Tx = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

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it is clear that T is closed but not continuous.

* Theorem:- let X and Y be two n/s and T is transformation from X into Y, Then T is closed iff $x_n \to x$ in X, $Tx_n \to y \Rightarrow y = Tx$.

Note:- A transformation T is closed iff its graph is closed set in $X \times Y$.

Proof:- Necessary Condition:-
let T be a closed transformation from X into Y. Then its graph G(T) will be closed in $X \times Y$.
let $x_n \to x$ in X, $Tx_n \to y$. Then it is clear that $(x_n, Tx_n) \to (x, y)$
since $(x_n, Tx_n) \in G(T)$, for (x,y), $\exists$ a sequence $\{(x_n, Tx_n)\}$ in G(T) such that $(x_n, Tx_n) \to (x,y)$
$\Rightarrow (x,y)$ is limit point of G(T).
$\Rightarrow (x,y) \in G(T)$ $\{\because G(T)$ is closed$\}$
$\Rightarrow y = Tx$ $\{$by def^n of $G(T) = (x,y) \in X \times Y$ $\{y = Tx\}\}$
Hence, condition is necessary.

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Sufficient Condition:
let $x_n \to x$ in X, $Tx_n \to y$ in Y implies $y \in Tx$.
Then we shall show that T is closed in $X \times Y$ i.e. G(T) is closed subset of $X \times Y$.
let (x,y) be a limit point of G(T), then $\exists$ a sequence $\{(x_n, Tx_n)\}$ in G(T) s.t. $(x_n, Tx_n) \to (x,y)$.
$\Rightarrow x_n \to x$ in X, and $Tx_n \to y$ in Y then $y = Tx$ (by assumption)
$\Rightarrow (x,y) \in G(T)$ i.e. G(T) contains all it limit point.
$\Rightarrow$ G(T) is closed.
$\Rightarrow$ T is closed. Hence Proved.